Improved Bound on Vertex Degree Version of Erdős Matching Conjecture

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Definitions and Notations

- For a set S and an integer $k \geq 1$, $\binom{S}{k} = \{e \subseteq S \mid |e| = k\}$;
- For an integer $n \geq 1$, $[n] = \{1, 2, \ldots, n\}$;

Notations

- A *hypergraph* H consists of a vertex set $V(H)$ and an edge set $E(H)$ whose members are subsets of $V(H)$. H is *k-uniform* if $E(H) \subseteq \binom{V(H)}{k}$. It is also called a *$k$-graph*.

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- A *perfect matching* in H is a matching of H that covers all the vertices of H.

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- A *fractional matching* in a k-graph $H = (V, E)$ is a function $f : E \to [0, 1]$ of weights of edges, such that for each $v \in V$ we have

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- A fractional matching f of H is *maximum* if $\sum_{e \in E} f(e) \geq \sum_{e \in E} g(e)$ for any fractional matching g of H.
- f is a *fractional perfect matching* if it has size $|V|/k$.

Complexity

- Fractional Matching Problem is a Linear Programming Problem; so it is a P-problem;
- Matching Problem in 2-graph is P-problem;
  Tutte's Theorem, Gallai-Edmonds Structure Theorem...
- When $k \geq 3$, Matching Problem in k-graphs is NPC.

Dirac's Theorem

- It is natural to study **degree** conditions that guarantee a **perfect matching (or near perfect matching, or fractional perfect matching or rainbow matching or stability)** in k-graphs **(or** l-**partite** k**-graphs, where** $k \leq l$**)**

- The size of a maximum matching in regular k-graphs.

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- For $r \in \{0, 1, ..., k-1\}$ and $S \in \binom{V(H)}{r}$, the *neighborhood* of S in H is denoted by $N_H(S) := \{U \in \binom{V(H)-S}{k-r} : S \cup U \in E(H)\}$. The *degree* of S is $d_H(S) := |N_H(S)|$.

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- The *minimum r-degree* of H, denoted by $\delta_r(H)$, is

$$\min\{d_H(S) \mid S \in \binom{V(H)}{r}\}.$$

$r = k-1$: minimum *co-degree* of H.

$r = 1$: minimum *vertex degree*.

$r = 0$: $\delta_0(H) = |E(H)|$.

Conjecture and Progress

Conjecture (Erdös, 1965)

Let $n \geq \max\{ks, 2k+1\}$. *Let* H *be a* k*-graph with vertex set* $[n]$. *If*

$$e(H) > \max\{\binom{n}{k} - \binom{n-s+1}{k}, \binom{ks-1}{k}\},$$

then $\nu(H) \geq s$ *(also* $\nu_f(H) \geq s$*).*

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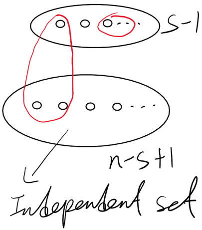
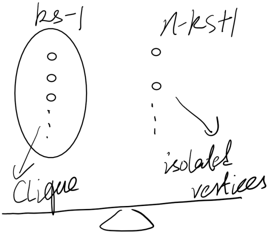




Erdös' Conjecture and Progress

- $s = 2$ (EKR Theorem, 1961);
- $k = 2$ (Erdös and Gallai, 1959)
- $k = 3$ and $n \geq 4t$ (Frankl, Rödl and Ruciński, CPC, 2012)
- $k = 3$ and n large (Luczak and Mieczkowska, JCTA 2014)
- $k = 3$ and all n (Frankl, DAM, 2017)
- $k = 3$, short proof (Frankl, Rödl and Ruciński, Acta Math. Hungar., 2017)

- $n \geq 2k^3 s$ (Bollobás, Daykin and Erdös, 1976)
- $n \geq 3k^2 s$ (Huang, Loh and Sudakov, CPC 2012)
- $n \geq (2s+1)k - s$ (Frankl, JCTA 2013)
  Stability version(Frankl and Kupavskii, JCTB 2019)
- $n \geq 5ks/3 - 2s/3$ and $s \geq s_0$ for large s_0 (Frankl and Kupavskii, 2018+)

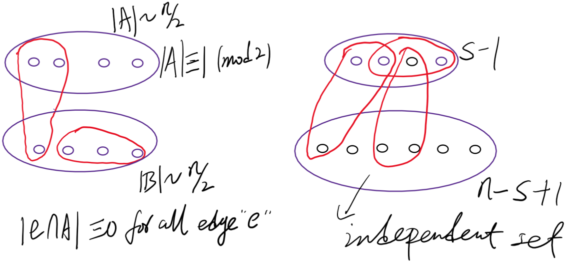
Conjecture and Progress

Conjecture (Hán, Person, Schacht, 2009; Kuhn and Osthus, 2009)

Let $n \equiv 0 \pmod{k}$*, and* $1 \le d \le k-1$*. Let* H *be a* k*-graph with vertex set* $[n]$*. If*

$$\delta_d(H) > \left(\max\{\frac{1}{2}, 1 - (\frac{k-1}{k})^{k-d}\} + o(1)\right)\binom{n-d}{k-d},$$

then H *has a perfect matching (also fractional perfect matching).*



Conjectures and Progress: Asymptotically tight bound

Let $m_d^s(k,n)$ (or $f_d^s(k,n)$) denote the minimum integer m such that every k-graph H on n vertices with $\delta_d(H) \geq m$ has a (fractional, resp.) matching of size s. Write $f_d(k,n) = c^* \binom{n-d}{k-d}$.

1. $k=3, d=1$, nearly tight (Han, Person and Schacht, SIAM 2009)
2. $$m_d^{n/k}(k,n) \sim \left(\max\{\frac{1}{2}, c^*\} + o(1)\right)\binom{n-d}{k-d},$$
   $(d,k) \in \{(1,4),\ (2,5),\ (1,5),\ (2,6) \text{ and } (3,7)\}$. (Alon et.al., JTCA, 2012)
3. $m_d^{n/k}(k,n) \leq (\frac{k-d}{k} + o(1))\binom{n-d}{k-d}$ (Hán, Person, Schacht, 2009).
4. $m_d^{n/k}(k,n) \leq (\frac{k-d}{k} - \frac{1}{k^{k-d}} + o(1))\binom{n-d}{k-d}$ (Markström and Ruciński, 2011)
5. $m_d^{n/k}(k,n) \leq (\frac{k-d}{k} - \frac{k-d-1}{k^{k-d}} + o(1))\binom{n-d}{k-d}$ (Kuhn, Osthus and Townsend, 2014)

Conjectures and Progress: Tight bound

1. $d = k - 1$ (Rödl, Ruciński and Szemerédi, JCTA 2009)
2. $d > k/2$ (Treglown and Zhao; JCTA 2012, 2013)
3. $k = 3, d = 1$ (Kuhn, Osthus and Treglown, JCTB 2013; Khan, SIAM 2013)
4. $k = 4, d = 1$ (Khan, JCTB 2016);
5. $d = 1,\ s = 2$ (Huang and Zhao, JCTA 2017)

Conjecture and Progress

Conjecture (Kuhn, Osthus and Townsend, 2014)

Let $n \equiv 0 \pmod{k}$, $n > mk$ *and* $1 \le d \le k-1$. *Let* H *be a* k*-graph with vertex set* $[n]$. *If*

$$\delta_d(H) > \binom{n-d}{k-d} - \binom{n-m+1-d}{k-d},$$

then H *has a matching of size* m.

1. $d = 1, n \ge 2k^3 s$ (Bollobás, Daykin and Erdös, 1976)
2. $d = 1, n \ge 3k^2 s$ (Huang and Zhao, JCTA 2017)
3. $d = k-1$ (Han, CPC 2016)
4. $d = k-2$ and $n \ne 1 \pmod{k}$ (Lu, Yu and Yuan, SIAM, 2021)
5. $d > k/2$ and $m < n/k - k^2$ (Lu, Yu and Yuan, SIAM, 2021)

Conjecture and Progress: Asymptotically tight bound

- $m_d^s(k,n) \sim \left(1-(1-s/n)^{k-d}\right)\binom{n-d}{k-d}$ for $d \geq k/2$ (Kühn, Osthus, and Townsend, EJC, 2014)
- $m_d^s(k,n) \sim \left(1-(1-s/n)^{k-d}\right)\binom{n-d}{k-d}$ for $d \geq 0.42k$ (Han, SIAM, 2017)
- $m_d^s(k,n) \sim \left(1-(1-s/n)^{k-d}\right)\binom{n-d}{k-d}$ for $d \geq 0.40k$ (Lu and Yu, 2018+)

Conjecture and Progress: our result

Theorem [Guo, Lu and Jiang, 2020+]

Let n, m and k be three integers such that $k \geq 3$, $n \geq 2km$ and n is sufficiently large. Let H be a k-graph on n vertices. If $\delta_1(H) > \binom{n-1}{k-1} - \binom{n-m}{k-1}$, then $\nu(H) \geq m$.

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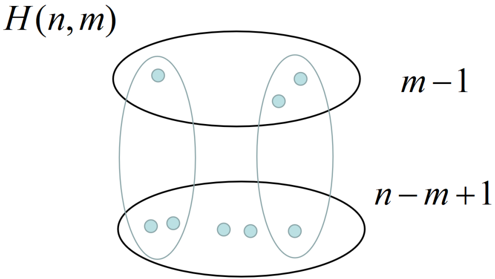

Proof Sketch

Definition

If $|E(H(n,m)) - E(H)| \leq \varepsilon n^{k+1}$, then we call $\mathcal{F}$ is ε-close to $H(n,m)$.

Case 1 H is ε-close to extremal graph $\mathcal{H}(n,m)$

Case 2 H is not ε-close to extremal graph $H(n,m)$

Proof Sketch-Case 1

Let $\varepsilon \ll c \ll 1/k$ and $n - km > cn$.

1. If H has a vertex v of degree at least $\binom{n-1}{k-1} - \binom{n-km-1}{k-1}$, it is sufficient to show that $H - v$ has a matching of size $m - 1$;
2. Else if $\Delta(H) < \binom{n-1}{k-1} - \binom{n-km-1}{k-1}$, then we have

$$|E(H(n,m)) \setminus E(H)| > \varepsilon n^k,$$

a contradiction.

Proof Sketch-Case 2

Construction

Let $0 < \alpha \ll \varepsilon$ and let $t = (\frac{1}{k-1} - \alpha)(n - km)$. Let $Q = \{v_1, \dots, v_t\}$. Let $\mathcal{H}$ be a k-graph with vertex set $Q \cup [n]$ and edge set

$$E(\mathcal{H}) = E(H) \cup \{e \in \binom{Q \cup [n]}{k} \mid e \cap Q \neq \emptyset\}.$$

Proof Sketch-Case 2

For completing Step 2, we need the following lemma.

Lemma (Frankl and Rödl, 1985)

For any γ, k, there exist large D and τ such that the following result holds. Every k-graph on n vertices with

$$(1-\tau)D < d_G(v) < (1+\tau)D \text{ for all } v \in V(G)$$

and

$$d_G(\{x,y\}) < \tau D \text{ for any two vertices } x, y \in V(G)$$

contains a matching covering all but at most γn vertices.

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So we need to show that $\mathcal{H}$ has a spanning subgraph F such that $(1-\tau)D < d_F(v) < (1+\tau)D$ and $d_F(\{x,y\}) < \tau D$.

Proof Sketch

Let $h : E(\mathcal{H}) \to [0,1]$ such that

$$\sum_{v \in e} h(e) \sim D \text{ for all } v \in V(\mathcal{H})$$

and

$$\sum_{\{x,y\} \subseteq e \in E(\mathcal{H})} h(e) \leq o(D) \quad \text{for any pair } x, y \in V(\mathcal{H}).$$

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Randomly choose edge e with probability $h(e)$, the resulted random graph F satisfies:

$$\mathbb{E} d_F(v) \sim (1 + o(1))D \text{ and } \mathbb{E} d_F(\{x,y\}) \le o(D)$$

for all $v, x, y \in V(\mathcal{F})$.

Proof Sketch: how to find such function h

Observation

If we may find $r = n/\ln n$ fractional perfect matchings $f_1, \dots, f_r$ such that

$$\sum_{i=1}^{r} \sum_{\{x,y\} \subseteq e} f_i(e) \leq 2 \quad \text{for any } \{x,y\} \in \binom{V(H)}{2}, \tag{1}$$

then $h = \sum_{i=1}^{r} f_i$ is a desired function.

Han-Kohayakawa-Person: Greedily Strategy

Suppose that we have $f_1, \dots, f_s$, where $s < r$. If for $\{x,y\} \in \binom{V(\mathcal{H})}{2}$, $\sum_{i=1}^{r} \sum_{\{x,y\} \subseteq e} f_i(e) > 2$, then we delete all edges containing $\{x,y\}$.

Proof Sketch: how to find such function h

Write $\psi_s = \sum_{i=1}^{s} f_s$. Define

$$A_s = \{\{x, y\} \in \binom{V(\mathcal{H})}{2} \mid \sum_{\{x,y\} \subseteq e} \psi_s(e) \geq 2\}$$

Let G be a graph with vertex set $V(\mathcal{H})$ and edge set A_s. Since $\sum_{x \in e} \psi_s(e) = s < n/\ln n$, then $\triangle(G) \leq (k-1)s < (k-1)n/\ln n$.

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Let

$$E_s = \{e \in E(\mathcal{H}) \mid \exists\{x,y\} \in A_s \text{ s.t. } \{x,y\} \subseteq e\}.$$

Then

$$\triangle(\mathcal{H} - E_s) \geq \binom{n-1}{k-1} - \binom{n-m}{k-1} - ((k-1)n/\ln n) * n^{k-2}.$$

Finding f_1 – More Definitions

Definition of Fractional Vertex Cover

Let $\omega : V(G) \to [0,1]$ such that $\sum_{x \in e} \omega(x) \geq 1$ for all $e \in E(G)$. Then ω is called a fractional vertex cover

Minimum Fractional Vertex Cover

ω is called minimum fractional vertex cover if $\sum_{e \in E(G)} \omega(e) \leq \sum_{e \in E(G)} \omega'(e)$ for any fractional cover ω'.

$\sum_{x \in V(G)} \omega(x)$ is called the size of vertex cover ω.

Let $vc(G)$ denote the size of minimum fractional vertex cover of G.

Proof Sketch – Finding f_1

1. Let $\omega : V(\mathcal{H}) \to [0,1]$ be a minimum fractional vertex cover of $\mathcal{H}$ such that $\omega(v_1) \geq \cdots \geq \omega(v_t)$ for $1 \leq i \leq t$ and $\omega(1) \geq \cdots \geq \omega(n)$.

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2. Let $Cl(\mathcal{H})$ be a k-graph with vertex set $V(\mathcal{H}) = [n] \cup Q$ and edge set

$$E(Cl(\mathcal{H})) = \{S \in \binom{Q \cup [n]}{k} \mid \sum_{x \in S} \omega(x) \geq 1\}.$$

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3. $\mathcal{H}$ is a subgraph of $Cl(\mathcal{H})$ and

$$\sum_{v \in V(\mathcal{H})} \omega(v) = vc(\mathcal{H}) = \nu_f(\mathcal{H})$$

$$\leq \nu_f(Cl(\mathcal{H}) = vc(Cl(\mathcal{H}))) \leq \sum_{v \in V(\mathcal{H})} \omega(v).$$

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4. So it is sufficient to show that $Cl(\mathcal{H})$ has a (fractional) perfect matching.

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4. $N_{Cl(\mathcal{H})}(\{n\})$ is close to $H(n+t, m+t)$;
5. We greedily find a matching M_1. Then we find a matching M_2 of size $\frac{1}{k-1}(n+t) - |M_1|$ in $N_{Cl(H)-V(M_1)}(n)$ and so obtain a matching M_2' of $Cl(H) - V(M_1) - \{n\}$;

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6. We extend $M_1 \cup M_2'$ into a perfect matching of $\mathcal{H}$.

Proof Sketch – Finding f_1

1. $N_{Cl(\mathcal{H})}(\{n\}) \subseteq N_{Cl(\mathcal{H})}(\{i\})$ for $i \in [n]$;
2. If $N_{Cl(\mathcal{H})}(\{n\}) - Q$ has a matching of size $\frac{1}{k}(n-(k-1)t)$, then $Cl(\mathcal{H}) - Q$ has a matching M of size $\frac{1}{k}(n-(k-1)t)$; we may extend M into a perfect matching of $Cl(\mathcal{H})$.
3. If $N_{Cl(\mathcal{H})}(\{n\}) - Q$ contains no matching of size $\frac{1}{k}(n-(k-1)t)$, then $N_{Cl(\mathcal{H})}(\{n\}) - Q$ is close to $H(n,m)$; [Lu, Yu and Yuan]
4. $N_{Cl(\mathcal{H})}(\{n\})$ is close to $H(n+t, m+t)$;
5. We greedily find a matching M_1. Then we find a matching M_2 of size $\frac{1}{k-1}(n+t) - |M_1|$ in $N_{Cl(H)-V(M_1)}(n)$ and so obtain a matching M_2' of $Cl(H) - V(M_1) - \{n\}$;
6. We extend $M_1 \cup M_2'$ into a perfect matching of $\mathcal{H}$.
7. Thus $\mathcal{H}$ has a fractional perfect matching.

Thanks for your attention!